

On knots and links in lens spaces

Alessia Cattabriga, Enrico Manfredi, Michele Mulazzani

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Abstract

In this paper we study some aspects of knots and links in lens spaces. Namely, if we consider lens spaces as quotient of the unit ball B^3 with suitable identification of boundary points, then we can project the links on the equatorial disk of B^3 , obtaining a regular diagram for them. In this contest, we obtain a complete finite set of Reidemeister type moves establishing equivalence, up to ambient isotopy, a Wirtinger type presentation for the fundamental group of the complement of the link and a diagrammatic method giving the first homology group. We also compute Alexander polynomial and twisted Alexander polynomials of this class of links, showing their correlation with Reidemeister torsion.

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1 Introduction

Knot theory is a widespread branch of geometric topology, with many applications to theoretical physics, chemistry and biology. The mainstream of this research have been concentrated for more than one century in the study of knots/links in the 3-sphere, which is the simplest closed 3-manifolds, and where the theory is completely equivalent to the one in the familiar space R^3 . That study was mainly conducted by the use of regular diagrams, which

are suitable projection of the knot/link in a disk/plane. In this way the 3-dimensional equivalence problem is translated in a 2-dimensional equivalence problem of diagrams. Reidemeister proved that two knots/links are equivalent if any of their diagrams can be connected by a finite sequence of three local moves, called Reidemeister moves. Diagrams also help to obtain invariants as the fundamental group of the exterior of the link (also called group of the link), via Wirtinger theorem, while the homology groups, as well as higher homotopy groups, are not relevant in the theory. From the fundamental group other important invariants as Alexander polynomials (classical and twisted) have been obtained, while from the diagram state sum type invariants derive, as Jones polynomials and quandle invariants.

In the last two decades, studies on knots/links have been generalized in more complicated spaces as solid torus (see [Be], [Ga1], [Ga2]), or lens spaces, which are the simplest closed 3-manifolds different from the 3-sphere. Particularly important are the class of $(1, 1)$ -knots (knots in either \mathbf{S}^3 or a lens space, also called genus one 1-bridge knots) intensively studied by many authors (see [CM], [CK], [Fu], [Ha], [MS], [Wu]).

In 1991, Drobotukhina introduced diagrams and moves for knots and links in the projective space, which is a special case of lens space, obtaining in this way an approach to compute a Jones type invariant for these links (see [D]). More recently, Huynh and Le in [HL] obtained a formula for the computation of the twisted Alexander polynomial for links in the projective space.

In this paper we extend some of those results for knots/links in the whole family of lens spaces. Our approach uses the model of lens spaces obtained by suitable identification on the boundary of a 3-ball described in Section 2, where a concept of regular projection and relative diagrams for the link is defined. In Section 3 we show that the equivalence between links in lens spaces can be translated in equivalence between diagrams, via a finite sequence of seven types of moves, generalizing the Reidemeister ones. In Section 4 a Wirtinger type presentation for the group of the link is given. In this context the homology group are not abelian free groups (as in \mathbf{S}^3), since a torsion part appears, and in Section 5 a method to compute that directly from the diagram is given. In Section 6 we deal with the twisted Alexander polynomials of these links, finding different properties and exploiting the connection with the Reidemeister torsion.

2 Diagrams

In this paper we work in the *Diff* category (of smooth manifolds and smooth maps). Every result also holds in the *PL* category, and in the *Top* category if we consider only tame links.

A *link* L in a closed 3-manifold M^3 is a 1-dimensional submanifold $L \subset M^3$. Obviously, L is homeomorphic to ν copies of \mathbf{S}^1 . When $\nu = 1$ the link is called a *knot*. Two links $L', L'' \subset M^3$ are called *equivalent* if there exists an ambient isotopy $H : M^3 \times [0, 1] \rightarrow M^3$ such that $h_1(L') = L''$, where $h_t(x) = H(x, t)$.

Consider the unit ball $B^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 \leq 1\}$ and let E_+ and E_- be respectively the upper and the lower closed hemisphere of ∂B^3 . Call B_0^2 the equatorial disk, defined by the intersection of the plane $x_3 = 0$ with B^3 , and label with N and S respectively the "north pole" $(0, 0, 1)$ and the "south pole" $(0, 0, -1)$ of B^3 .

If p and q are two coprime integers such that $0 \leq q < p$, let $g_{p,q} : E_+ \rightarrow E_+$ be the rotation of $2\pi q/p$ around the x_3 -axis, as in Figure 1, and $f_3 : E_+ \rightarrow E_-$ be the reflection with respect to the plane $x_3 = 0$. The *lens space* $L(p, q)$ is the quotient of B^3 by the equivalence relation on ∂B^3 which identifies $x \in E_+$ with $f_3 \circ g_{p,q}(x) \in E_-$. We denote by $F : B^3 \rightarrow L(p, q) = B^3 / \sim$ the quotient map. Note that on the equator $\partial B_0^2 = E_+ \cap E_-$ each equivalence class contains p points.

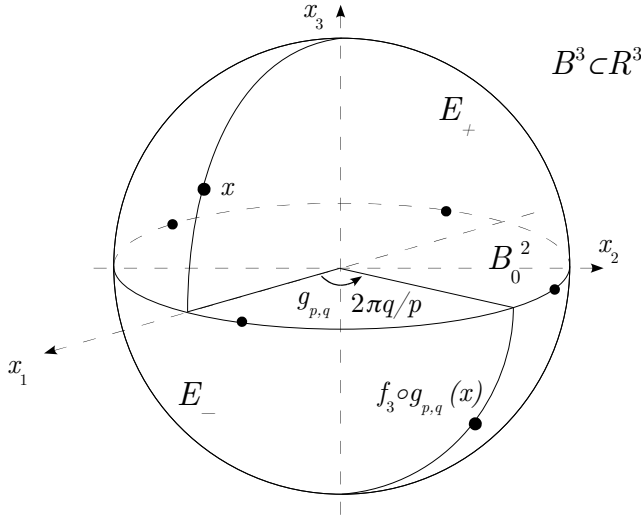


Figure 1: Representation of $L(p, q)$.

It is easy to see that $L(1, 0) \cong \mathbf{S}^3$ since $g_{1,0} = \text{Id}_{E_+}$. Furthermore, $L(2, 1)$ is \mathbb{RP}^3 , since the above construction gives the usual model of the projective

space where opposite points on the boundary of B^3 are identified.

In the following we improve the definition of diagram for links in lens spaces given by Gonzato [G]. Assume $p > 1$, since $L(1,0) \cong \mathbf{S}^3$ is the classical case. Let L be a link in $L(p,q)$ and consider $L' = F^{-1}(L)$. By moving L via a small isotopy in $L(p,q)$, we can suppose that:

- i) L' does not meet the poles N and S of B^3 ;
- ii) $L' \cap \partial B^3$ consists of a finite set of points;
- iii) L' is not tangent to ∂B^3 ;
- iv) $L' \cap \partial B_0^2 = \emptyset$.¹

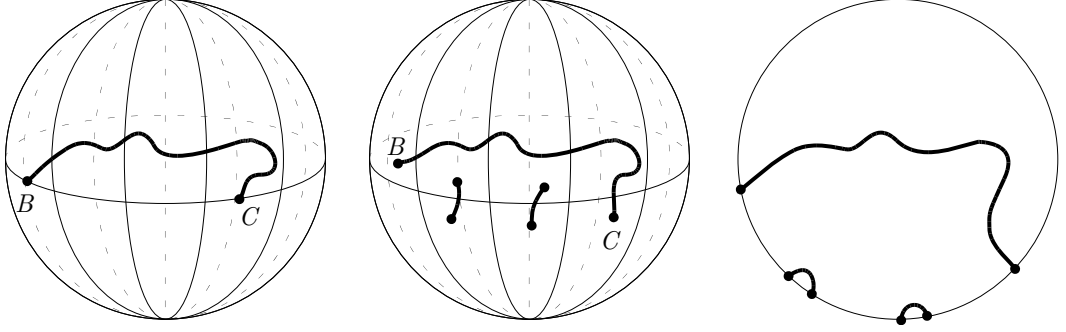


Figure 2: Avoiding ∂B_0^2 in $L(9,1)$.

As a consequence, L' is the disjoint union of closed curves in $\text{int} B^3$ and arcs properly embedded in B^3 (i.e., only the boundary points belong onto ∂B^3).

Let $\mathbf{p} : B^3 \setminus \{N, S\} \rightarrow B_0^2$ be the projection defined by $\mathbf{p}(x) = c(x) \cap B_0^2$, where $c(x)$ is the circle (possibly a line) through N , x and S . Take L' and project it using $\mathbf{p}|_{L'} : L' \rightarrow B_0^2$. For $P \in \mathbf{p}(L')$, the set $\mathbf{p}|_{L'}^{-1}(P)$ may contain more than one point; in this case, we say that P is a *multiple point*. In particular, if it contains exactly two points, we say that P is a *double point*. We can assume, by moving L via a small isotopy, that the projection $\mathbf{p}|_{L'} : L' \rightarrow B_0^2$ of L is *regular*, namely:

- 1) the projection of L' contains no cusps;
- 2) all auto-intersections of $\mathbf{p}(L')$ are transversal;

¹The small isotopy that allows L' to avoid the equator ∂B_0^2 is depicted in Figure 2.

- 3) the set of multiple points is finite, and all of them are actually double points;
- 4) no double point is on ∂B_0^2 .

Now let Q be a double point, consider $\mathbf{p}_{|L'}^{-1}(Q) = \{P_1, P_2\}$ and suppose that P_1 is closer to N than P_2 . Let U be a connected open neighborhood of P_2 in L' such that $\mathbf{p}(\bar{U})$ contains no other double point and does not meet ∂B_0^2 . We call U *underpass* relative to Q . Every connected component of the complement in L' of all the underpasses (as well as its projection in B_0^2) is called *overpass*.

A *diagram* of a link L in $L(p, q)$ is a regular projection of $L' = F^{-1}(L)$ on the equatorial disk B_0^2 , with specified overpasses and underpasses² (see Figure 3).

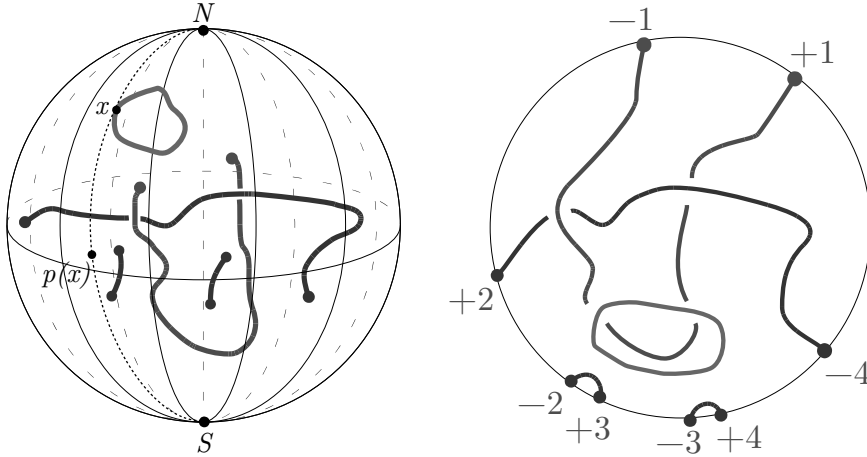


Figure 3: A link in $L(9, 1)$ and the corresponding diagram.

We assume that the equator is oriented counterclockwise if we look at it from N . According to the orientation, label with $+1, \dots, +t$ the endpoints of the overpasses belonging to the upper hemisphere, and with $-1, \dots, -t$ the endpoints on the lower hemisphere, respecting the rule $+i \sim -i$. An example is shown in Figure 3.

Note that for the case $L(2, 1) \cong \mathbb{RP}^3$ we get exactly the diagram described in [D].

²As usual, the projections of the underpasses are not depicted in the diagram.

3 Generalized Reidemeister moves

In this section we obtain a finite set of moves connecting two different diagrams of the same link. The *generalized Reidemeister moves* on a diagram of a link $L \subset L(p, q)$, are the moves $R_1, R_2, R_3, R_4, R_5, R_6$ and R_7 of Figure 4. Observe that, when $p = 2$ the moves R_5 and R_6 are equal, and R_7 is a trivial move.

Theorem 1. *Two links L_0 and L_1 in $L(p, q)$ are equivalent if and only if their diagrams can be joined by a finite sequence of generalized Reidemeister moves R_1, \dots, R_7 and diagram isotopies, when $p > 2$. If $p = 2$, moves R_1, \dots, R_5 are sufficient.*

Proof. It is easy to see that each Reidemeister move connects equivalent links, hence a finite sequence of Reidemeister moves and diagram isotopies does not change the equivalence class of the link.

On the other hand, if we have two equivalent links L_0 and L_1 , then there exists an isotopy of the ambient space $H : L(p, q) \times [0, 1] \rightarrow L(p, q)$ such that $h_1(L_0) = L_1$. For each $t \in [0, 1]$ we have a link $L_t = h_t(L_0)$.

The link L_t may violate conditions i), ii), iii), iv) and its projection can violate the regularity conditions 1), 2), 3) and 4).

It is easy to see that the isotopy H can be chosen in such a way that conditions i) and ii) are satisfied at any time. Moreover, using general position theory (see [R] for details) we can assume that there are a finite number of forbidden configurations and that for each $t \in [0, 1]$, only one of them may occur. The remaining conditions might be violated during the isotopy as depicted in the left part of Figure 4. More precisely,

- conditions 1), 2) and 3) generate configurations V_1, V_2 and V_3 ;
- condition iii) generates V_4 ;
- condition 4) generates V_5 and V_6 ; the difference between the two configurations is that V_5 involves two arcs of L' ending in the same hemisphere of ∂B^3 , while V_6 involves arcs ending in different hemispheres;
- from condition iv) we have a family of configurations $V_{7,1}, \dots, V_{7,p-1}$ (see Figure 5); the difference between them is that $V_{7,1}$ has the endpoints of the projection identified directly by $g_{p,q}$, while $V_{7,k}$ has the endpoints identified by $g_{p,q}^k$, for $k = 2, \dots, p - 1$.

From each type of forbidden configuration a transformation of the diagram appears, i.e. a generalized Reidemeister move, as follows (see Figure 4):

- from V_1, V_2 and V_3 we obtain the usual Reidemeister moves R_1, R_2 and R_3 ;

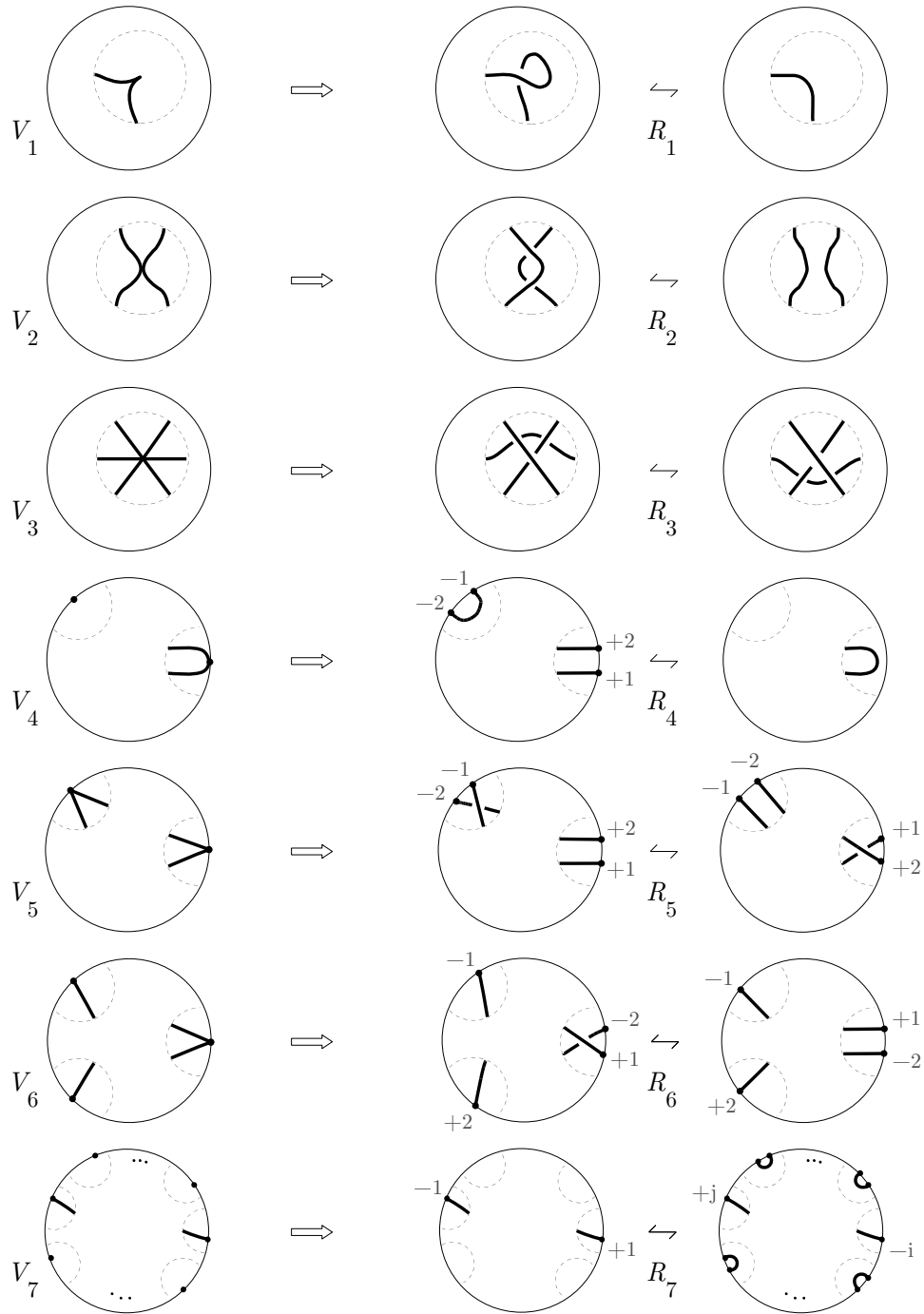


Figure 4: Generalized Reidemeister moves.

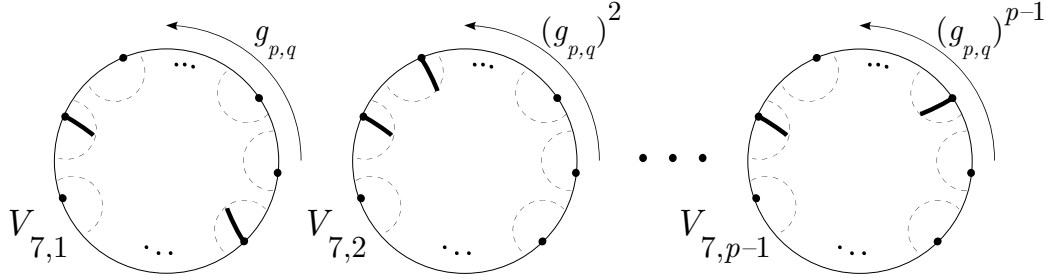


Figure 5: Forbidden configurations $V_{7,1}, V_{7,2}, \dots, V_{7,p-1}$.

- from V_4 we obtain move R_4 ;
- from V_5 , we obtain two different moves: R_5 if the overpasses endpoints belong to the same hemisphere, and R_6 otherwise;
- from $V_{7,1}, \dots, V_{7,p-1}$ we obtain the moves $R_{7,1}, \dots, R_{7,p-1}$.

Nevertheless the moves $R_{7,2}, \dots, R_{7,p-1}$ can be seen as the composition of $R_7 = R_{7,1}, R_6, R_4$ and R_1 moves. More precisely, the move $R_{7,k}$, with $k = 2, \dots, p-1$, is obtained by the following sequence of moves: first we perform an R_7 move on the two overpasses corresponding to the points $+i$ and $-i$, then we repeat $k-1$ times the three moves R_6 - R_4 - R_1 necessary to retract the small arc having the endpoints with the same sign (see an example in Figure 6).

So we can drop out $R_{7,2}, \dots, R_{7,p-1}$ from the set of moves and keep only $R_{7,1} = R_7$. As a consequence, any pair of diagrams of two equivalent links can be joined by a finite sequence of generalized Reidemeister moves R_1, \dots, R_7 and diagram isotopies. When $p = 2$, it is easy to see that R_6 coincides with R_5 , and R_7 is a trivial move; so in this case moves R_1, \dots, R_5 are sufficient (see also [D]). \square

Diagram isotopies have to respect the identifications of boundary points of the link projection. Therefore, move R_6 is possible only if there are no other arcs inside the small circles of the move R_6 , as depicted in Figure 4. For example, Figure 7 shows the case of a link in $L(3, 1)$ where the R_6 move removing the crossing cannot be performed.

4 Fundamental group

In this section we obtain, directly from the diagram, a finite presentation for the fundamental group of the complement of links in $L(p, q)$.

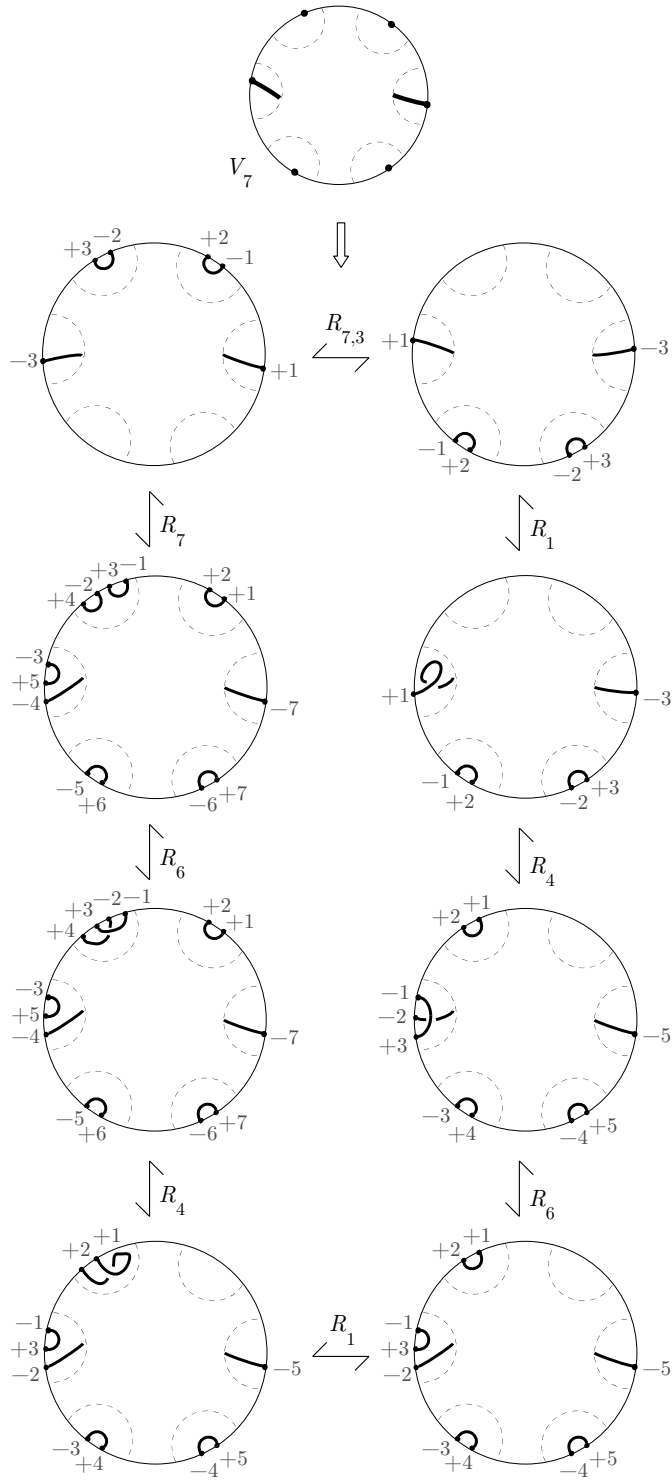


Figure 6: How to decompose a move $R_{7,3}$.

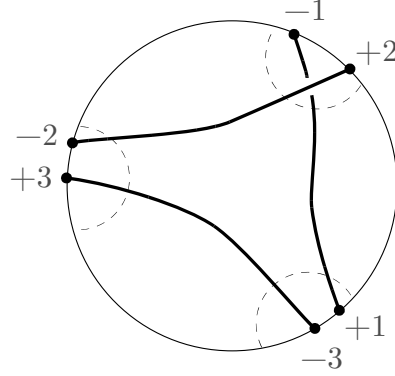


Figure 7: A forbidden R_6 move.

Let L be a link in $L(p, q)$, and consider a diagram of L . Fix an orientation for L , which induces an orientation on both L' and $\mathbf{p}(L')$. Perform an R_1 move on each overpass of the diagram having both endpoints on the boundary of the disk; in this way every overpass has at most one boundary point. Then label the overpasses as follows: A_1, \dots, A_t are the ones ending in the upper hemisphere, namely in $+1, \dots, +t$, while A_{t+1}, \dots, A_{2t} are the overpasses ending in $-1, \dots, -t$. The remaining overpasses are labelled by A_{2t+1}, \dots, A_r . For each $i = 1 \dots, t$, let $\epsilon_i = +1$ if, according to the link orientation, the overpass A_i starts from the point $+i$; otherwise, if A_i ends in the point $+i$, let $\epsilon_i = -1$.

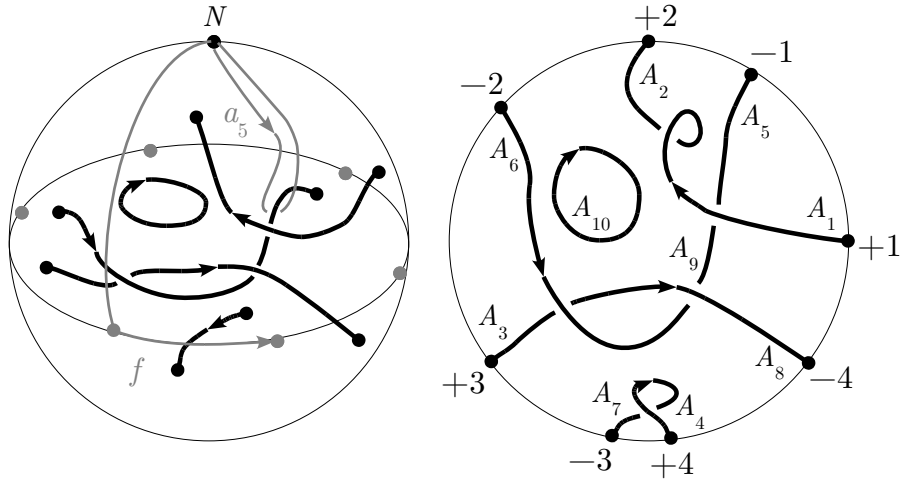


Figure 8: Example of overpasses labelling for a link in $L(6, 1)$.

Associate to each overpass A_i a generator a_i , which is a loop around the overpass as in the classical Wirtinger theorem, oriented following the left hand rule. Moreover let f be the generator of the fundamental group of the lens space depicted in Figure 8. The relations are the following:

W: w_1, \dots, w_s are the classical Wirtinger relations for each crossing, that is to say $a_i a_j a_i^{-1} a_k^{-1} = 1$ or $a_i a_j^{-1} a_i^{-1} a_k = 1$, according to Figure 9;

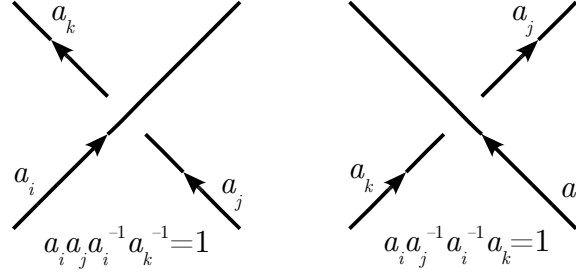


Figure 9: Wirtinger relations.

L: l is the lens relation $a_1^{\epsilon_1} \dots a_t^{\epsilon_t} = f^p$;

M: m_1, \dots, m_t are relations (of conjugation) between loops corresponding to overpasses with identified endpoints on the boundary. If $t = 1$ the relation is $a_2^{\epsilon_1} = a_1^{-\epsilon_1} f^q a_1^{\epsilon_1} f^{-q}$. Otherwise, consider the point $-i$ and, according to equator orientation, let $+j$ and $+j + 1 \pmod{t}$ be the type $+$ points aside of it. We distinguish two cases:

- if $-i$ lies on the diagram between -1 and $+1$, then the relation m_i is

$$a_{t+i}^{\epsilon_i} = \left(\prod_{k=1}^j a_k^{\epsilon_k} \right)^{-1} f^q \left(\prod_{k=1}^{i-1} a_k^{\epsilon_k} \right) a_i^{\epsilon_i} \left(\prod_{k=1}^{i-1} a_k^{\epsilon_k} \right)^{-1} f^{-q} \left(\prod_{k=1}^j a_k^{\epsilon_k} \right);$$

- otherwise, the relation m_i is

$$a_{t+i}^{\epsilon_i} = \left(\prod_{k=1}^j a_k^{\epsilon_k} \right)^{-1} f^{q-p} \left(\prod_{k=1}^{i-1} a_k^{\epsilon_k} \right) a_i^{\epsilon_i} \left(\prod_{k=1}^{i-1} a_k^{\epsilon_k} \right)^{-1} f^{p-q} \left(\prod_{k=1}^j a_k^{\epsilon_k} \right).$$

Theorem 2. Let $* = F(N)$, then the group of the link $L \subset L(p, q)$ is:

$$\pi_1(L(p, q) \setminus L, *) = \langle a_1, \dots, a_r, f \mid w_1, \dots, w_s, l, m_1, \dots, m_t \rangle.$$

Proof. Suppose that $L' = F^{-1}(L)$ is such that $\mathbf{p}_{|L'} : L' \rightarrow B_0^2$ is a regular projection. Consider a sphere \mathbf{S}_ε^2 of radius $1 - \varepsilon$, with $0 < \varepsilon < 1$; this sphere splits the 3-ball B^3 into two parts: call B_ε^3 the internal one and E_ε the external one. Choose ε small enough such that all the underpasses belong into $\text{int}(B_\varepsilon^3)$. Let N_ε be the north pole of B_ε^3 , and consider $\tilde{\mathbf{S}}_\varepsilon^2 = \mathbf{S}_\varepsilon^2 \cup \overline{NN_\varepsilon}$.

In order to compute $\pi_1(L(p, q) \setminus L, *)$, we apply Seifert-Van Kampen theorem with decomposition $(L(p, q) \setminus L) = (F(\tilde{B}_\varepsilon^3) \setminus L) \cup (F(E_\varepsilon) \setminus L)$.

The fundamental group of $F(\tilde{B}_\varepsilon^3) \setminus L$ can be obtained as in the classical Wirtinger Theorem:

$$\pi_1(F(\tilde{B}_\varepsilon^3) \setminus L, *) = \langle a_1, \dots, a_r \mid w_1, \dots, w_s \rangle.$$

For $F(E_\varepsilon) \setminus L$, we proceed in the following way: first of all observe that we can retract $F(E_\varepsilon) \setminus L$ to $E \setminus L$, where E is $\partial B^3 / \sim$. According to the orientation, fix a point T_1 in ∂B_0^2 just before $+1$ and such that its equivalent points T_2, \dots, T_p (via \sim) do not belong to $\mathbf{p}(L')$. Following the example

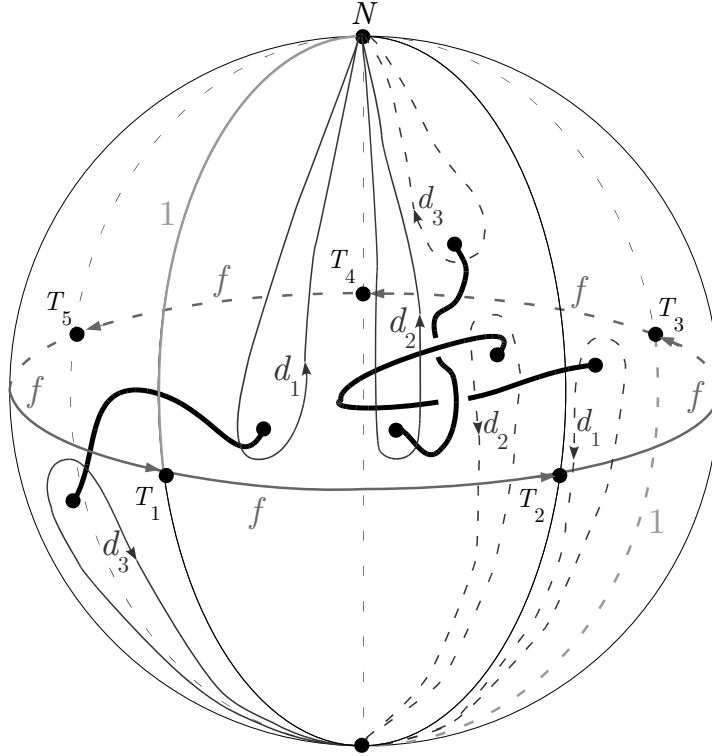


Figure 10: Boundary complex for a knot in $L(5, 2)$.

of Figure 10, the 2-complex E is a CW-complex composed by: two 0-cells

$N = S$ and $T_1 = T_2 = \dots = T_p$, two 1-cells $\widehat{NT_1}$ (chosen as a maximal tree in the 1-skeleton) and $\widehat{T_1T_2}$ (corresponding to f), and one 2-cell, that is the upper hemisphere. In order to obtain $\pi_1(E \setminus L, *)$, we need to add the loops d_1, \dots, d_t around the points of L . The relation given by the 2-simplex is $d_1 \cdots d_t = f^p$. Hence the fundamental group of $E \setminus L$ is:

$$\pi_1(E \setminus L, *) = \langle d_1, \dots, d_t, f \mid d_1 \cdots d_t = f^p \rangle. \quad (1)$$

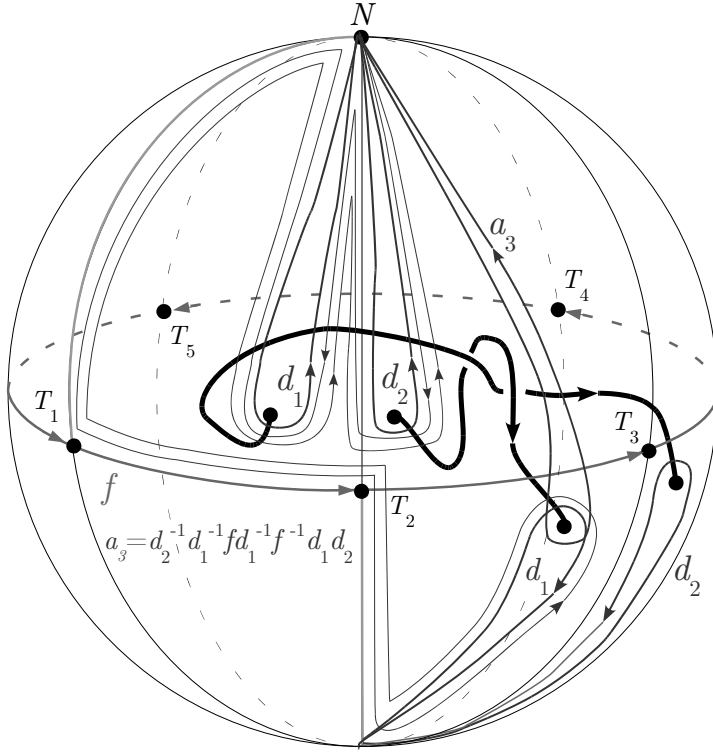


Figure 11: Example of relation for a link in $L(5, 1)$.

Finally, the fundamental group of $F(\tilde{S}_\varepsilon^2) \setminus L = (F(\tilde{B}_\varepsilon^3) \setminus L) \cap (F(E_\varepsilon) \setminus L)$ is generated by a_1, \dots, a_{2t} . By Seifert-Van Kampen theorem, we identify each a_1, \dots, a_t with the corresponding generator d_1, \dots, d_t , according to orientation: $a_i^{\epsilon_i} = d_i$. Furthermore we need to identify a_{t+1}, \dots, a_{2t} with suitable loops in the CW-complex, distinguishing two cases:

- if $-i$ lies on the diagram between -1 and $+1$, then we obtain the following relation (see Figure 11 for an example)

$$a_{t+i}^{\epsilon_i} = \left(\prod_{k=1}^j d_k \right)^{-1} f^q \left(\prod_{k=1}^{i-1} d_k \right) d_i \left(\prod_{k=1}^{i-1} d_k \right)^{-1} f^{-q} \left(\prod_{k=1}^j d_k \right);$$

- otherwise, the relation is

$$a_{t+i}^{\epsilon_i} = \left(\prod_{k=1}^j d_k \right)^{-1} f^{q-p} \left(\prod_{k=1}^{i-1} d_k \right) d_i \left(\prod_{k=1}^{i-1} d_k \right)^{-1} f^{p-q} \left(\prod_{k=1}^j d_k \right).$$

At last we remove d_1, \dots, d_t from the group presentation, obtaining:

$$\pi_1(L(p, q) \setminus L, *) = \langle a_1, \dots, a_r, f \mid w_1, \dots, w_s, l, m_1, \dots, m_t \rangle. \quad \square$$

In the special case of $L(2, 1) = \mathbb{RP}^3$, the presentation is equivalent (via Tietze transformations) to the one given in [HL].

Remark 3. If the link diagram does not contain overpasses which are circles (we can avoid this case by using suitable R_1 moves), then the presentation of Theorem 2 is balanced (i.e., the number of generators equals the number of relations). Indeed, it is enough to think at each intersection between the diagram and the boundary disk as a fake crossing. Moreover, the product of the Wirtinger relators represents a loop that is trivial in $\pi_1(E \setminus L, *)$, so anyone of the Wirtinger relations can be deduced from the others, obtaining a presentation of deficiency one.

5 First homology group

In this section we show how to determine, directly from the diagram, the first homology group of links in $L(p, q)$, which is useful for the computation of twisted Alexander polynomials.

Consider a diagram of an oriented knot $K \subset L(p, q)$ and let ϵ_i be as defined in the previous section. If $n_1 = |\{\epsilon_i \mid \epsilon_i = +1, i = 1, \dots, t\}|$ and $n_2 = |\{\epsilon_i \mid \epsilon_i = -1, i = 1, \dots, t\}|$, define $\delta_K = q(n_2 - n_1) \bmod p$.

Lemma 4. *If $K \subset L(p, q)$ is an oriented knot and $[K]$ is the homology class of K in $H_1(L(p, q))$, then $[K] = \delta_K$.*

Proof. Let f be the generator of $H_1(L(p, q)) = \mathbb{Z}_p$, as depicted in Figure 12. Let $K \cap (\partial B^3 / \sim) = \{P_1, \dots, P_t\}$. For $i = 1, \dots, t$, consider the identification class $[P_i]_{\sim} = \{P'_i, P''_i\}$, with $P'_i \in E_+$ and $P''_i \in E_-$. Denote with γ_i the path (actually a loop in $L(p, q)$) connecting P'_i with P''_i as in Figure 12, oriented as depicted if $\epsilon_i = +1$ and in the opposite direction if $\epsilon_i = -1$. Of course its homology class is $[\gamma_i] = \epsilon_i q$. The loop $K' = K \cup \gamma_1 \cup \dots \cup \gamma_t$ is homologically trivial, so we have: $0 = [K'] = [K] + \sum_{i=1}^t [\gamma_i] = [K] + (n_1 - n_2)q$, and therefore $[K] = \delta_K$. \square

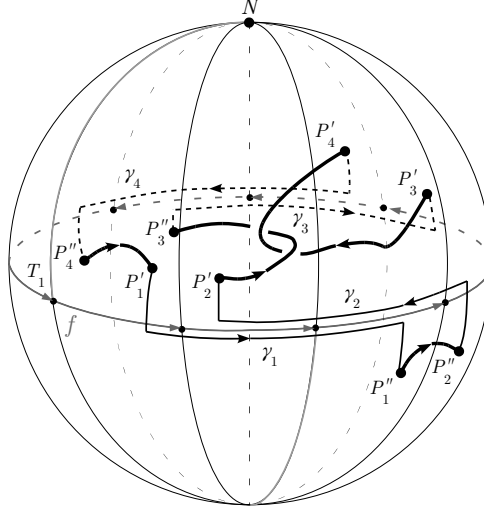


Figure 12: Equatorial arcs for a knot in $L(7, 2)$.

Corollary 5. *Let L be a link in $L(p, q)$, with components L_1, \dots, L_ν . For each $j = 1, \dots, \nu$, let $\delta_j = [L_j] \in \mathbb{Z}_p = H_1(L(p, q))$. Then*

$$H_1(L(p, q) \setminus L) \cong \mathbb{Z}^\nu \oplus \mathbb{Z}_d,$$

where $d = \gcd(\delta_1, \dots, \delta_\nu, p)$.

Proof. We abelianize the fundamental group presentation given in Section 4. Relations of type W and M imply that generators corresponding to the same link component are homologous. So $H_1(L(p, q) \setminus L)$ is generated by g_1, \dots, g_ν , which are generators corresponding to the link components, and f . Relation L becomes: $pf - (\tilde{\delta}_1 g_1 + \dots + \tilde{\delta}_\nu g_\nu) = 0$, with $\tilde{\delta}_j = \sum_{A_h \subset L_j} \epsilon_h$, where L_j is the j -th component of L . Therefore $H_1(L(p, q) \setminus L) \cong \mathbb{Z}^\nu \oplus \mathbb{Z}_d$, where $d = \gcd(\tilde{\delta}_1, \dots, \tilde{\delta}_\nu, p)$. Since $\gcd(p, q) = 1$ and, by Lemma 4, $\delta_j = -q\tilde{\delta}_j$, we obtain $d = \gcd(\tilde{\delta}_1, \dots, \tilde{\delta}_\nu, p) = \gcd(\delta_1, \dots, \delta_\nu, p)$. \square

6 Twisted Alexander polynomials

In this section we analyze the twisted Alexander polynomials of links in lens spaces and their relationship with Reidemeister torsion. Start by recalling the definition of twisted Alexander polynomials (for further references see [T]). Given a finitely generated group π , denote with $H = \pi/\pi'$ its abelianiza-

tion and let $G = H/\text{Tors}(H)$. Take a presentation $\pi = \langle x_1, \dots, x_m \mid r_1 \dots, r_n \rangle$ and consider the Alexander-Fox matrix A associated to the presentation, that is $A_{ij} = \text{pr}(\frac{\partial r_i}{\partial x_j})$, where pr is the natural projection $\mathbb{Z}[F(x_1, \dots, x_m)] \rightarrow \mathbb{Z}[\pi] \rightarrow \mathbb{Z}[H]$ and $\frac{\partial r_i}{\partial x_j}$ is the Fox derivative of r_i . Moreover let $E(\pi)$ be the first elementary ideal of π , which is the ideal of $\mathbb{Z}[H]$ generated by the $(m-1)$ -minors of A . For each homomorphism $\sigma : \text{Tors}(H) \rightarrow \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ we can define a twisted Alexander polynomial $\Delta^\sigma(\pi)$ of π as follows: fix a splitting $H = \text{Tors}(H) \times G$ and consider the ring homomorphism that we still denote with $\sigma : \mathbb{Z}[H] \rightarrow \mathbb{C}[G]$ sending (f, g) , with $f \in \text{Tors}(H)$ and $g \in G$, to $\sigma(f)g$, where $\sigma(f) \in \mathbb{C}^*$. The ring $\mathbb{C}[G]$ is a unique factorization domain and we set $\Delta^\sigma(\pi) = \text{gcd}(\sigma(E(\pi)))$. This is an element of $\mathbb{C}[G]$ defined up to multiplication by elements of G and non-zero complex numbers. If $\Delta(\pi)$ denote the classic Alexander polynomial we have $\Delta^1(\pi) = \alpha \Delta(\pi)$, with $\alpha \in \mathbb{C}^*$.

If $L \subset L(p, q)$ is a link in a lens space then the σ -twisted Alexander polynomial of L is $\Delta_L^\sigma = \Delta^\sigma(\pi_1(L(p, q) \setminus L))$. Since in this case $\text{Tors}(H) = \mathbb{Z}_d$ then $\sigma(\text{Tors}(H))$ is contained in the cyclic group generated by ζ , where ζ is a d -th primitive root of the unity. When $\mathbb{Z}[\zeta]$ is a principal ideal domain, in order to define Δ_L^σ we can consider the restriction $\sigma : \mathbb{Z}[H] \rightarrow \mathbb{Z}[\zeta][G]$. Note that $\Delta_L^\sigma \in \mathbb{Z}[\zeta][G]$ is defined up to multiplication by $\zeta^h g$, with $g \in G$. In this setting we recall the following theorem.

Proposition 6. [MM] *If ζ is a d -th primitive root of unity, then the ring $\mathbb{Z}[\zeta]$ is a principal ideal domain if and only if $d \cong 2 \pmod{4}$ or d is one of the following 30 integers: 1, 3, 4, 5, 7, 8, 9, 11, 12, 13, 15, 16, 17, 19, 20, 21, 24, 25, 27, 28, 32, 33, 35, 36, 40, 44, 45, 48, 60, 84.*

A link is called *local* if it is contained in a ball embedded in $L(p, q)$. For local links the following properties hold.

Proposition 7. *Let L be a local link in $L(p, q)$. Then $\Delta_L^\sigma = 0$ if $\sigma \neq 1$, and $\Delta_L = p \cdot \Delta_{\bar{L}}$ otherwise, where \bar{L} is the link L considered as a link in \mathbb{S}^3 .*

Proof. The fundamental group of L can be presented with the relations of Wirtinger type and the lens relation $f^p = 1$ only. Therefore the column in the Alexander-Fox matrix A corresponding to the Fox derivative of the lens relation is everywhere zero except for the entry corresponding to the f -derivative, which is $1 + f + f^2 + \dots + f^{p-1}$. Moreover, the cofactor of this non-zero entry is equal to the Alexander-Fox matrix of \bar{L} . So the statement follows by observing that in the case of Δ_L , the generator f is sent to 1, while if $\sigma \neq 1$, the generator f is sent in a k -th root of the unity, where k divides p , and so $\sigma(1 + f + f^2 + \dots + f^{p-1}) = 0$. \square

As a consequence a knot with a non trivial twisted Alexander polynomial cannot be local.

Figure 13 shows the twisted Alexander polynomials of a local trefoil knot in $L(4, 1)$ and proves that twisted Alexander polynomial may distinguish knots with the same Alexander polynomial.

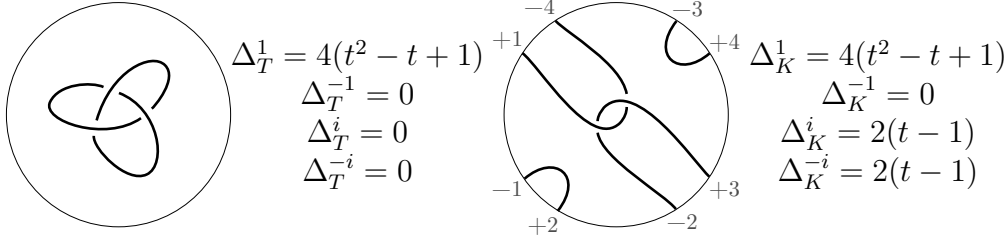


Figure 13: Twisted Alexander polynomials for two knots in $L(4, 1)$.

Let $L = L_1 \sharp L_2$, where \sharp denote the connected sum and L_2 is a local link. The decomposition $(L(p, q), L) = (L(p, q), L_1) \sharp (\mathbf{S}^3, L_2)$ induces monomorphisms $j_1 : H_1(L(p, q) \setminus L_1) \rightarrow H_1(L(p, q) \setminus L)$ and $j_2 : H_1(\mathbf{S}^3 \setminus L_2) \rightarrow H_1(L(p, q) \setminus L)$. Given $\sigma : \mathbb{Z}[H_1(L(p, q) \setminus L)] \rightarrow \mathbb{C}[G]$ induced by $\sigma \in \text{hom}(\text{Tors}(H_1(L(p, q) \setminus L)), \mathbb{C}^*)$, denote with σ_1 and σ_2 its restrictions to $\mathbb{Z}[j_1(H_1(L(p, q) \setminus L_1))]$ and $\mathbb{Z}[j_2(H_1(\mathbf{S}^3 \setminus L_2))]$ respectively. We have the following result.

Proposition 8. *Let $L = L_1 \sharp L_2 \subset L(p, q)$, where L_2 is local link. With the above notations we have $\Delta_L^\sigma = \Delta_{L_1}^{\sigma_1} \cdot \Delta_{L_2}^{\sigma_2}$.*

Proof. Since $(L(p, q), L) = (L(p, q), L_1) \sharp (\mathbf{S}^3, L_2)$, by Van Kampen theorem we get $\pi_1(L(p, q) \setminus L) = \langle a_1, \dots, a_n, b_1, \dots, b_m \mid r_1, \dots, r_{n-1}, s_1, \dots, s_{m-1}, a_1 = b_1 \rangle$, where $\pi_1(L(p, q) \setminus L_1, *) = \langle a_1, \dots, a_n \mid r_1, \dots, r_{n-1} \rangle$ and $\pi_1(\mathbf{S}^3 \setminus L_2, *) = \langle b_1, \dots, b_m \mid s_1, \dots, s_{m-1} \rangle$. So the Alexander-Fox matrix of L is

$$A_L = \begin{pmatrix} A_{L_1} & 0 \\ 0 & A_{L_2} \\ -1 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \end{pmatrix},$$

where A_{L_i} is the Alexander-Fox matrix of L_i , for $i = 1, 2$. If $d_k(A)$ denotes the greatest common division of all k -minors of a matrix A , then a simple computation shows that $d_{m+n-1}(A_L) = d_{n-1}(A_{L_1}) \cdot d_{m-1}(A_{L_2})$. Therefore it is easy to see that $\Delta_L^\sigma = \Delta_{L_1}^{\sigma_1} \cdot \Delta_{L_2}^{\sigma_2}$. \square

In Figure 14 we compute the twisted Alexander polynomials of the connected sum of a local trefoil knot \bar{T} with the three knots $K_0, K_1, K_2 \subset L(4, 1)$

depicted in the left part of the figure, respectively. Note that for the case of $K_2\sharp\overline{T}$, the map σ_2 , that is the restriction of σ to $\mathbb{Z}[j_2(H_1(\mathbf{S}^3 \setminus \overline{T}))]$, sends the generator $g \in \mathbb{Z}[H_1(S^3 \setminus \overline{T})]$ in $t^2 \in \mathbb{Z}[H_1(L(p, q) \setminus K_2\sharp\overline{T})]$ (resp. in $-t^2$) if $\sigma = 1$ (resp. if $\sigma = -1$), instead of t as it does for the classical Alexander polynomial.

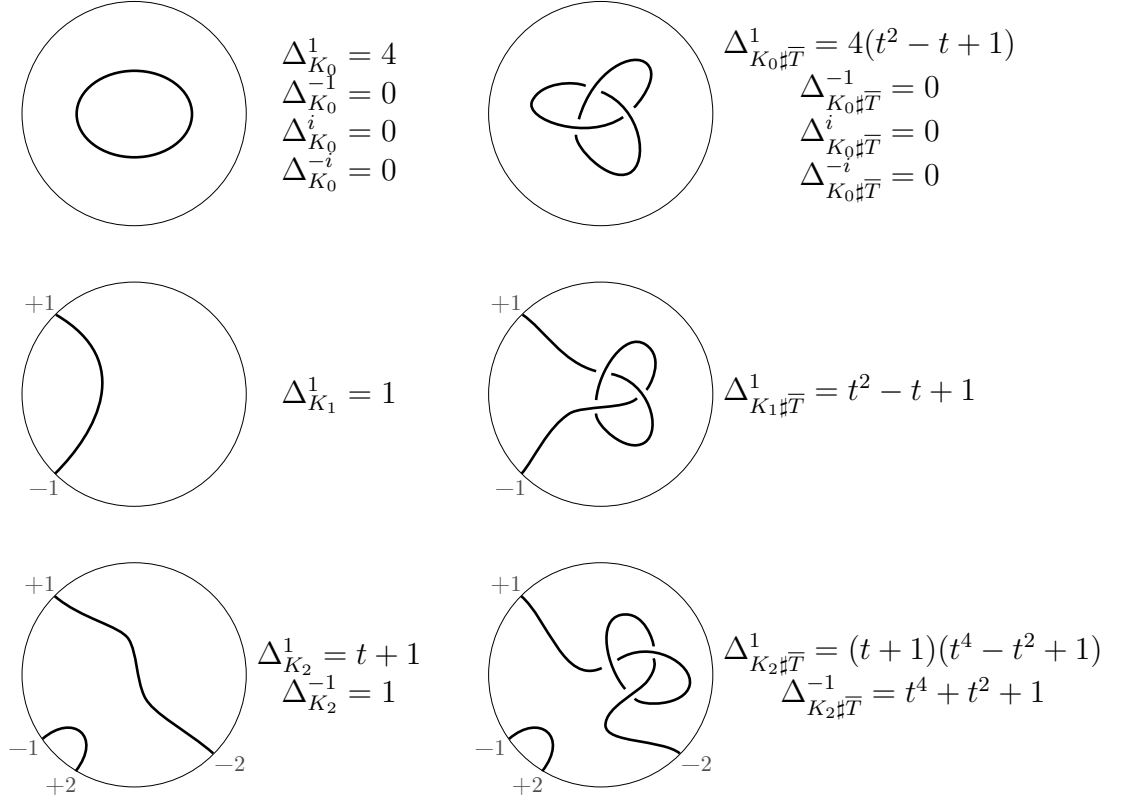


Figure 14: Twisted Alexander polynomials for three knots in $L(4, 1)$.

Proposition 9. [T] *Let L be a knot in a lens space then:*

- 1) $\Delta_L^\sigma(t) = \Delta_L^\sigma(t^{-1})$ (i.e., the twisted Alexander polynomial is symmetric);
- 2) $\Delta(1) = |\text{Tors}(H_1(L(p, q) \setminus L))|$.

Before giving the relationship between the twisted Alexander polynomials and the Reidemeister torsion we briefly recall the definition of Reidemeister torsion (for further references see [T]).

If c and c' are two basis of a finite-dimensional vector space over a field \mathbb{F} , denote with $[c/c']$ the determinant of the matrix whose columns are the

coordinates of the elements of c respect to c' . Let C be a finite chain complex of vector spaces

$$0 \rightarrow C_m \xrightarrow{\delta_m} C_{m-1} \xrightarrow{\delta_{m-1}} \cdots \xrightarrow{\delta_1} C_0 \rightarrow 0$$

which is acyclic (i.e., the sequence is exact) and based (i.e., a distinguished base is fixed for each vector space). For each $i \leq m$, let b_i be a sequence of vectors in C_i such that $\delta_i(b_i)$ is a base of $\text{Im}\delta_i$, and let c_i be the fixed base of C_i . The juxtaposition of $\delta_{i+1}(b_{i+1})$ and b_i gives a base of C_i denoted by $\delta_{i+1}(b_{i+1})b_i$. The torsion of C is defined as

$$\tau(C) = \prod_{i=0}^m [\delta_{i+1}(b_{i+1})b_i/c_i]^{(-1)^{i+1}} \in \mathbb{F}.$$

If C is not acyclic the torsion is defined to be zero.

For a finite connected CW-complex X , let $\pi = \pi_1(X)$ and $H = H_1(X) = \pi/\pi'$. Consider a ring homomorphism $\varphi : \mathbb{Z}[H] \rightarrow \mathbb{F}$ and let \hat{X} be the maximal abelian covering of X (corresponding to π'). Let $C_*(\hat{X})$ be the cellular chain complex associated to \hat{X} . Since H acts on \hat{X} via deck transformations, $C_*(\hat{X})$ is a complex of left $\mathbb{Z}[H]$ -modules. Moreover the homomorphism φ endows F with the structure of a $\mathbb{Z}[H]$ -module via $fz = f\varphi(z)$, with $f \in F$ and $z \in \mathbb{Z}[H]$. Then $\mathbb{F} \otimes_{\varphi} C_*(\hat{X})$ is a chain complex of finite dimensional vector spaces. The φ -torsion of X is defined to be $\tau(\mathbb{F} \otimes_{\varphi} C_*(\hat{X}))$. It depends on the choice of a base for $\mathbb{F} \otimes_{\varphi} C_*(\hat{X})$ and so the φ -torsion is defined up to multiplication by $\pm\varphi(h)$, with $h \in H$.

Let L be a link in $L(p, q)$ and let $X = L(p, q) \setminus L$, then X is homotopic to a 2-dimensional cell complex Y . The φ -torsion τ_L^{φ} of a link L is the φ -torsion of Y . In order to investigate the relationship between the torsion and the twisted Alexander polynomial, let $H = \text{Tors}(H) \times G$ and consider a map $\sigma : \mathbb{Z}[H] \rightarrow \mathbb{C}[G]$ associated to a certain $\sigma \in \text{hom}(\text{Tors}(H), \mathbb{C}^*)$, as described in the beginning of this section. If $\mathbb{C}(G)$ denotes the field of quotient of $\mathbb{C}[G]$, then by composing with the projection into the quotient, σ determines a homomorphism $\mathbb{Z}[H] \rightarrow \mathbb{C}(G)$ that we still denote with σ . In this way each $\sigma \in \text{hom}(\text{Tors}(H), \mathbb{C}^*)$ determines both a twisted Alexander polynomial Δ_L^{σ} and a torsion τ_L^{σ} .

We say that a link $L \subset L(p, q)$ is *nontorsion* if $\text{Tors}(H_1(L(p, q) \setminus L)) = 0$, otherwise we say that L is *torsion*. Note that a local link L in a lens space different from \mathbf{S}^3 is clearly torsion.

Theorem 10. *Let L be a link in $L(p, q)$. If L is a nontorsion knot and t is a generator of its first homology group, then $\tau_L^{\sigma}(t - 1) = \Delta_L^{\sigma}$. Otherwise $\tau_L^{\sigma}(t) = \Delta_L^{\sigma}$.*

Proof. According to Theorem 2 and Remark 3, the group $\pi_1(L(p, q) \setminus L)$ admits a presentation with m generators and $m-1$ relations. So, the Alexander-Fox matrix A associated to such a presentation is a $(m-1) \times m$ matrix. This means that $\Delta^\sigma(L) = \gcd(\sigma(A_1), \dots, \sigma(A_m))$, where A_i is the $(m-1)$ -minor of A obtained removing the i -th column. Let a_i be a generator of $\pi_1(L(p, q) \setminus L)$. The formula $(\sigma(a_i) - 1)\tau_L^\sigma = \det A_i$ that holds for links in the projective space (see [HL]) generalizes to lens spaces. So, in order to get the statement it is enough to prove that $\gcd(\sigma(a_1) - 1, \dots, \sigma(a_m) - 1)$ is equal to $t - 1$, where t is a generator of the free part of $H_1(L(p, q) \setminus L)$, if L is a torsion knot, and equal to 1 otherwise.

Let L be a torsion knot and denote with t and u a generator of the free part and the torsion part of $H_1(L(p, q) \setminus L)$ respectively. Moreover let d be the order of the torsion part of $H_1(L(p, q) \setminus L)$. If $\text{pr}(a_i) = t^{h_i}u^{n_i}$ then $\sigma(a_i) = t^{h_i}\zeta^{n_i}$ where ζ is a d -th root of the identity. A simple computation shows that g divides $t^{\sum_{i=1}^m h_i} \zeta^{\sum_{i=1}^m n_i} - 1$, for any $\alpha_i \in \mathbb{Z}$, where $g = \gcd(\sigma(a_1) - 1, \dots, \sigma(a_m) - 1)$. Since $t \in \text{pr}(\pi_1(L(p, q) \setminus L))$, there exist α_i such that $t = \prod_{i=1}^m \text{pr}(a_i^{\alpha_i}) = t^{\sum_{i=1}^m \alpha_i h_i} u^{\sum_{i=1}^m \alpha_i n_i}$; so $\sum_{i=1}^m \alpha_i h_i = 1$ and d divides $\sum_{i=1}^m \alpha_i n_i$. Then g divides $t - 1$ and therefore either $g = 1$ or $g = t - 1$. Analogously, since $u \in \text{pr}(\pi_1(L(p, q) \setminus L))$, there exists i_0 such that g divides $\sigma(a_{i_0}) - 1 = t^{h_{i_0}}\zeta^{n_{i_0}} - 1$ and n_{i_0} is not divided by d . The statement follows by observing that, in this case, $\gcd(t - 1, t^{h_{i_0}}\zeta^{n_{i_0}} - 1) = 1$.

If L is torsion and has at least two component then $\sigma(a_i) = t_1^{h_{i1}} \dots t_\nu^{h_{i\nu}} \zeta^{n_i}$, where ν is the number of components. The statement is obtained by setting $t_2 = \dots = t_\nu = 1$ and applying the previous argument to t_1 .

If L is a nontorsion knot, then $H_1(L(p, q) \setminus L) = \langle t \rangle$ and $\sigma(a_i) = t^{h_i}$. In this case it is easy to prove that $\gcd(t^{h_1} - 1, \dots, t^{h_m} - 1) = t - 1$.

Finally, if L is nontorsion and has at least two component, then $\sigma(a_i) = t_1^{h_{i1}} \dots t_\nu^{h_{i\nu}}$. By letting $t_j = 1$ for $j \neq i$ and applying the previous reasoning to t_i , for each $i = 1, \dots, \nu$, we obtain $\gcd(\sigma(a_1) - 1, \dots, \sigma(a_m) - 1) = \gcd(t_1 - 1, \dots, t_\nu - 1) = 1$. \square

These results generalize those obtained in [K] for knots in \mathbf{S}^3 and [HL] for link in $L(2, 1) \cong \mathbb{RP}^3$. Moreover, in [KL] an analogous result is obtained for CW-complexes but considering only a one-variable Alexander polynomial associated to an infinite cyclic covering of the complex.

If L has at least two components we can consider the projection $\varphi : \mathbb{Z}[\zeta][G] = \mathbb{Z}[\zeta][t_1, \dots, t_m, t_1^{-1}, \dots, t_m^{-1}] \rightarrow \mathbb{Z}[\zeta][t, t^{-1}]$, sending each variable t_i to t . The *one-variable* twisted Alexander polynomial of L is $\bar{\Delta}_L^\sigma = \varphi(\Delta_L^\sigma)$.

The same argument used in the previous proof leads to the following statement, regarding the one-variable twisted polynomial.

Theorem 11. *Let L be a link in $L(p, q)$ with at least two components. If L is a nontorsion link and t is a generator of its first homology group then $\tau_L^\sigma(t - 1) = \bar{\Delta}_L^\sigma$. Otherwise $\tau_L^\sigma(t) = \bar{\Delta}_L^\sigma$.*

The computation of $\bar{\Delta}_L^\sigma$ for knots in arbitrary lens spaces has been implemented in a program using Mathematica code: the input is a knot diagram in $L(p, q)$ given via a generalization of the Dowker-Thistlethwaite code (see [DT, DH, Ta]).

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ALESSIA CATTABRIGA, Department of Mathematics, University of Bologna, ITALY. E-mail: alessia.cattabriga@unibo.it

ENRICO MANFREDI, Department of Mathematics, University of Bologna, ITALY. E-mail: enrico.manfredi3@unibo.it

MICHELE MULAZZANI, Department of Mathematics and C.I.R.A.M.,
University of Bologna, ITALY. E-mail: michele.mulazzani@unibo.it